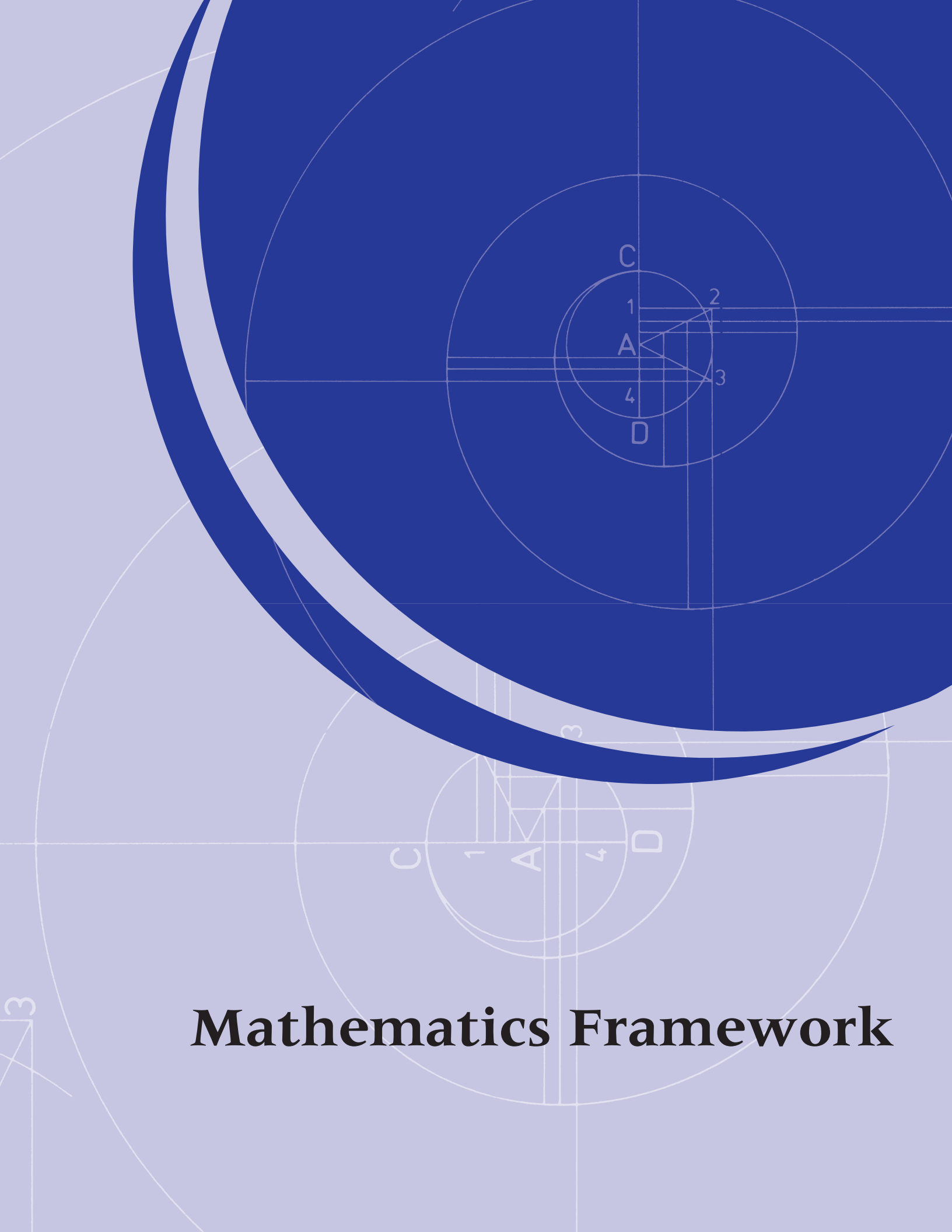
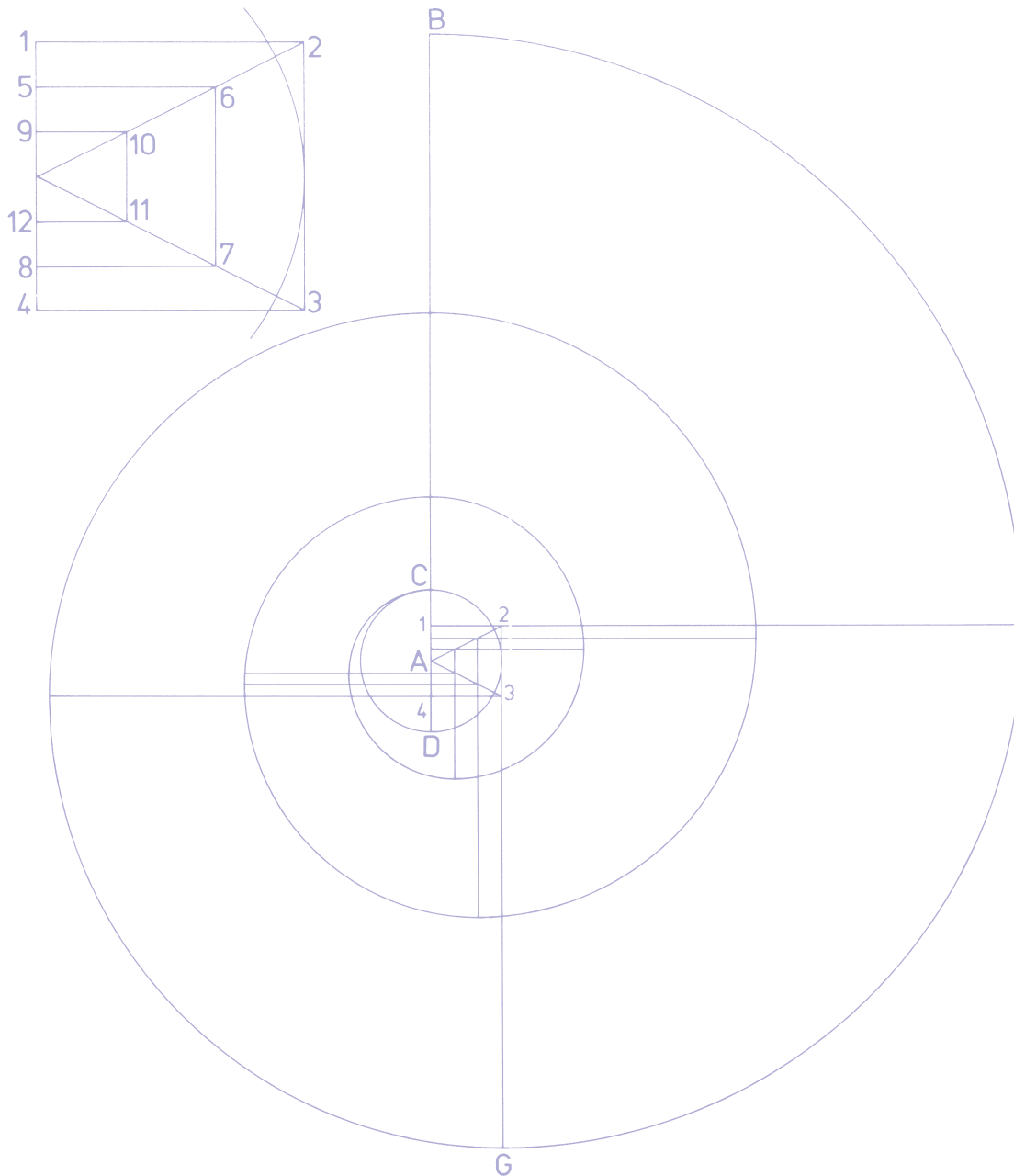




The background features a large, dark blue spiral that originates from a central point and expands outwards. Overlaid on this are several faint, light blue geometric diagrams. One prominent diagram is a circle with a horizontal diameter labeled 'C' at the left and 'D' at the right, and a vertical diameter labeled 'A' at the top and '4' at the bottom. A point '1' is marked on the vertical diameter between 'A' and '4'. A point '2' is marked on the circle's circumference in the upper right quadrant, and a point '3' is marked on the circumference in the lower right quadrant. Lines connect these points, forming a series of nested, overlapping shapes. Another similar diagram is visible below it, and a third one is partially visible in the bottom left corner.

Mathematics Framework



Mathematics Framework

Overview

The mathematics assessment framework for TIMSS 2003 is framed by two organizing dimensions, a content dimension and a cognitive dimension, analogous to those used in the earlier TIMSS assessments.¹ As outlined below, each dimension has several domains:

Mathematics Content Domains

-  Number
-  Algebra
-  Measurement
-  Geometry
-  Data

Mathematics Cognitive Domains

-  Knowing Facts and Procedures
-  Using Concepts
-  Solving Routine Problems
-  Reasoning

The two dimensions and their domains are the foundation of the mathematics assessment. The content domains define the specific mathematics subject matter covered by the assessment, and the cognitive domains define the sets of behaviors expected of students as they engage with the mathematics content. Each content domain has several topic areas (i.e., number is further categorized by whole numbers, fractions and decimals, integers, and ratio, proportion, and percent). Each topic area is presented as a list of objectives covered in a majority of participating countries, at either grade 4 or grade 8.²

Exhibit 2 shows the target percentages of testing time devoted to each content and cognitive domain for both the fourth and eighth grade assessments. The content and cognitive domains for the mathematics assessment are discussed in detail in the following sections. Example mathematics items and tasks are presented in Appendix B.

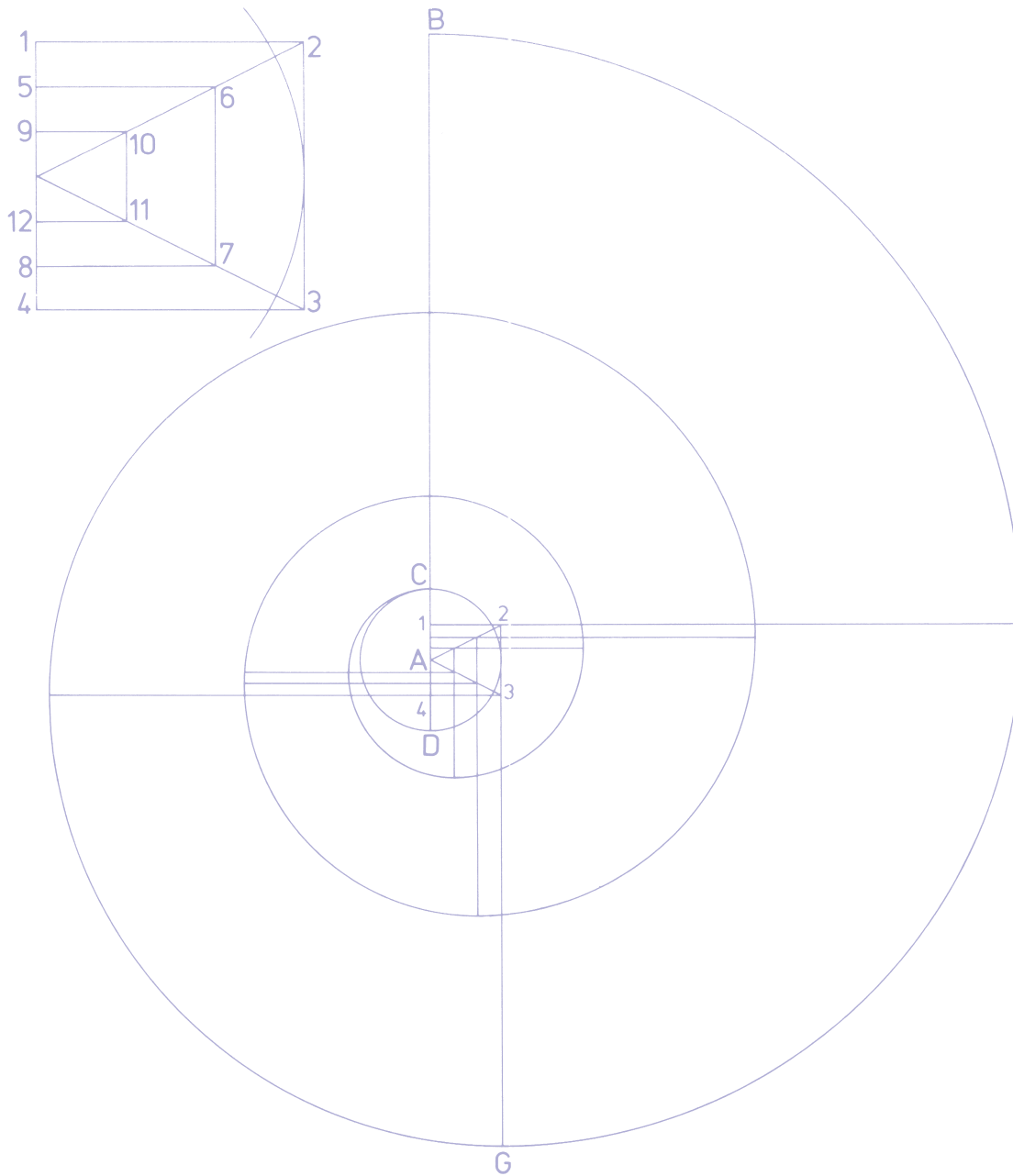
Exhibit 2: Target Percentages of TIMSS 2003 Mathematics Assessment Devoted to Content and Cognitive Domains by Grade Level

	Fourth Grade	Eighth Grade
Mathematics Content Domains		
Number	40%	30%
Algebra*	15%	25%
Measurement	20%	15%
Geometry	15%	15%
Data	10%	15%
Mathematics Cognitive Domains		
Knowing Facts and Procedures	20%	15%
Using Concepts	20%	20%
Solving Routine Problems	40%	40%
Reasoning	20%	25%

*At fourth grade, the Algebra content domain is called Patterns, Equations, and Relationships.

¹ Similarly, the curriculum frameworks for TIMSS 1995 and 1999 assessments included content areas and performance expectations (Robitaille, D.F., et al, (1993), *TIMSS Monograph No. 1: Curriculum Frameworks for Mathematics and Science*, Vancouver, BC: Pacific Educational Press).

² More information about the factors considered in finalizing the topics and assessment objectives is provided in the Introduction.



Mathematics Content Domains

As mentioned earlier, the five content domains described in the mathematics framework, with assessment objectives defined that are appropriate for either the fourth or eighth grade, are:

- Number
- Algebra
- Measurement
- Geometry
- Data

The structure of the content dimension of the TIMSS framework reflects the importance of being able to continue comparisons of achievement with previous assessments in these content domains. The organization of topics across the content domains reflects some minor revision in the definition of the reporting categories used in the 1995 and 1999 assessments, particularly for the fourth grade.³ The current structure, however, permits the direct mapping of trend items from 1995 and 1999 into the content domains defined for 2003. Thus, each mathematics content domain is considered as an analysis and reporting category.

The grade-specific assessment objectives indicated by topic areas within content domains define the assessment areas appropriate for each mathematics reporting category. These grade-specific objectives are written in terms of student understandings or abilities that items aligned with these objectives are designed to elicit. The range of behaviors assessed to measure student understandings and abilities is discussed in the section of the mathematics framework describing the cognitive domains. While the assessment of

abilities such as solving non-routine problems and reasoning with mathematics will be of special interest, the factual, procedural, and conceptual knowledge that form the initial base for the development and implementation of these skills will also be assessed.

Problem solving and communication are key outcomes of mathematics education that are associated with many of the topics in the content domains. They are regarded as valid behaviors to be elicited by test items in most topic areas.

The categorization by mathematics topic area within content domains at the fourth grade parallels that used at the eighth grade. Not all of the topic areas, however, are appropriate for the fourth grade. Also, the mathematical and cognitive levels of items developed according to the assessment objectives in the frameworks will be appropriate for the grade/age group. For example, at the fourth grade, there is a greater emphasis on number relative to the other major domains.

The following sections describe each of the mathematics content domains. They give an overview of the topic areas to be covered in the TIMSS assessment, focusing on the difference in student understandings expected at the fourth and eighth grades. Following the general description of each content domain is a table indicating a set of assessment outcomes for each main assessment topic area. These assessment outcomes are written in terms of behaviors to be elicited by items that exemplify the understandings and abilities expected of students at each grade level.

³ The five reporting categories used in the TIMSS 1999 international report for eighth grade were Fractions and Number Sense; Measurement; Data Representation, Analysis, and Probability; Geometry; and Algebra. The six reporting categories used in the TIMSS 1995 international report for fourth grade were Whole Numbers; Fractions and Proportionality; Measurement, Estimation, and Number Sense; Data Representation, Analysis, and Probability; Geometry; and Patterns, Relations, and Functions.

Number

The number content domain includes understanding of counting and numbers, ways of representing numbers, relationships among numbers, and number systems. At the fourth and eighth grades, students should have developed number sense and computational fluency, understand the meanings of operations and how they relate to one another, and be able to use numbers and operations to solve problems.

The number content domain consists of understandings and skills related to:

- Whole numbers
- Fractions and decimals
- Integers
- Ratio, proportion, and percent

In this domain, there is more emphasis on computing with whole numbers at the fourth grade than at the eighth grade. Since whole numbers provide the easiest introduction to operations with numbers that are basic to the development of mathematics, working with whole numbers is the foundation of mathematics in the primary school. Most children learn to count at a young age and can solve simple addition, subtraction, multiplication, and division

problems during the first few years of school. Grade 4 students should be able to compute with whole numbers of reasonable size, estimate the sums, differences, products, and quotients, and use computation to solve problems.

In the area of common fractions and decimal fractions, the emphasis is on representation and translation between forms, understanding what quantities the symbols represent, and computation and problem solving. At the fourth grade, students should be able to compare familiar fractions and decimals. By the eighth grade, they should be able to move flexibly among equivalent fractions, decimals, and percents using a range of strategies.

Although the integer topic area is not appropriate for grade 4, students in the middle grades of schooling should extend their mathematical understanding from whole numbers to integers, including order and magnitude as well as operations with integers.

Assessing students' ability to work with proportions is an important component. Aspects of proportional reasoning can include numerical and qualitative comparison problems as well as the more traditional missing value problems (i.e., presenting three values and asking students to find the fourth or missing value).

Number: Whole Numbers

Grade 4

- Represent whole numbers using words, diagrams, or symbols, including recognizing and writing numbers in expanded form.
- Demonstrate knowledge of place value.
- Compare and order whole numbers.
- Identify sets of numbers according to common properties such as odd and even, multiples, or factors.
- Compute with whole numbers.
- Estimate computations by approximating the numbers involved.
- Solve routine and non-routine problems, including real-life problems.

Grade 8

- Demonstrate knowledge of place value and of the four operations.
- Find and use factors or multiples of numbers, and identify prime numbers.
- Express in general terms and use the principles of commutativity, associativity, and distributivity.
- Evaluate powers of numbers, and square roots of perfect squares to 144.
- Solve problems by computing, estimating, or approximating.

Number: Fractions and Decimals

Grade 4

- Recognize fractions as parts of unit wholes, parts of a collection, locations on number lines, divisions of whole numbers.
- Identify equivalent fractions.
- Compare and order fractions.
- Show understanding of decimals.
- Represent fractions or decimals using words, numbers, or models.
- Add and subtract fractions with the same denominator.
- Add and subtract with decimals.

Notes: Grade 4 fractions items will involve denominators of 2, 3, 4, 5, 6, 8, 10, or 12.

Grade 4 decimals items will involve decimals to tenths and/or hundredths.

Grade 8

- Compare and order fractions.
- Compare and order decimals.
- Demonstrate knowledge of place value for decimals.
- Represent decimals and fractions using words, numbers, or models (including number lines).
- Recognize and write equivalent fractions.
- Convert fractions to decimals and vice versa.
- Relate operations with fractions or decimals to situations and models.
- Compute with fractions and decimals, including use of commutativity, associativity, and distributivity.
- Approximate decimals to estimate computations.
- Solve problems involving fractions.
- Solve problems involving decimals.

Number: Integers

Grade 4

- Not assessed.

Grade 8

- Represent integers using words, numbers, or models (including number lines).
- Compare and order integers.
- Show understanding of addition, subtraction, multiplication, and division with integers.
- Compute with integers.
- Solve problems using integers.

Number: Ratio, Proportion, and Percent

Grade 4

- Solve problems involving simple proportional reasoning.

Grade 8

- Identify and find equivalent ratios.
- Divide a quantity in a given ratio.
- Convert percents to fractions or decimals, and vice versa.
- Solve problems involving percents.
- Solve problems involving proportions.

Algebra

While functional relationships and their uses for modeling and problem solving are of prime interest, it is also important to assess how well the supporting knowledge and skills have been learned. The algebra content domain includes patterns and relationships among quantities, using algebraic symbols to represent mathematical situations, and developing fluency in producing equivalent expressions and solving linear equations.

Because algebra is generally not taught as a formal subject in primary school, this content domain will be identified as *Patterns, Equations, and Relationships* at the fourth grade. In contrast, at the eighth grade the algebra reporting category will reflect understandings across all of the topic areas below.

The major topic areas in algebra are:

- Patterns
- Algebraic expressions
- Equations and formulas
- Relationships

Students will be asked to recognize and extend patterns and relationships. They will also be asked to recognize and use symbols to represent situations algebraically. At the fourth grade, understandings related to patterns, simple equations, and the idea of functions as they apply to pairs of numbers are included. Algebraic concepts are more formalized at the eighth grade, and students should focus on understanding linear relationships and the concept of variable. Students at this level are expected to use and simplify algebraic formulas, solve linear equations and inequalities and pairs of simultaneous equations involving two variables, and use a range of linear and nonlinear functions. They should be able to solve real-world problems using algebraic models and to explain relationships involving algebraic concepts.

Algebra: Patterns

Grade 4

- Extend and find missing terms of numeric and geometric patterns.
- Match numeric and geometric patterns with descriptions.
- Describe relationships between adjacent terms in a sequence or between the number of the term and the term.

Grade 8

- Extend numeric, algebraic, and geometric patterns or sequences using words, symbols, or diagrams; find missing terms.
- Generalize pattern relationships in a sequence, or between adjacent terms, or between the number of the term and the term, using words or symbols.

Algebra: Algebraic Expressions

Grade 4

- Not assessed.

Grade 8

- Find sums, products, and powers of expressions containing variables.
- Evaluate expressions for given numeric values of the variable(s).
- Simplify or compare algebraic expressions to determine equivalence.
- Model situations using expressions.

Algebra: Equations and Formulas

Grade 4

- Show understanding of equality using equations, areas, volumes, masses/weights.
- Find the missing number in an equation (e.g., if $17 + \underline{\quad} = 29$, what number would go in the blank to make the equation true?).
- Model simple situations involving unknowns with an equation.
- Solve problems involving unknowns.

Grade 8

- Evaluate formulas given values of the variables.
- Use formulas to answer questions about given situations.
- Indicate whether a value, or values, satisfies a given equation.
- Solve simple linear equations and inequalities, and simultaneous (two variables) equations.
- Write linear equations, inequalities, or simultaneous equations that model given situations.
- Solve problems using equations or formulas.

Algebra: Relationships

Grade 4

- Generate pairs of numbers following a given rule (e.g., multiply the first number by 3 and add 2 to get the second number).
- Write, or select, a rule for a relationship given some pairs of numbers satisfying the relationship.
- Graph pairs of numbers following a given rule.
- Show why a pair of numbers follows a given rule. (E.g., a rule for a relation between two numbers is "multiply the first number by 5 and subtract 4 to get the second number." Show that when the first number is 2 and the second number is 6 the rule is followed.)

Grade 8

- Recognize equivalent representations of functions as ordered pairs, tables, graphs, words, or equations.
- Given a function in one representation, generate a different but equivalent representation.
- Recognize and interpret proportional, linear, and nonlinear relationships (travel graphs and simple piecewise functions included).
- Write or select a function to model a given situation.
- Given a graph of a function, identify attributes such as intercepts on axes, and intervals where the function increases, decreases, or is constant.



Measurement

Measurement involves assigning a numerical value to an attribute of an object. The focus of this content domain is on understanding measurable attributes and demonstrating familiarity with the units and processes used in measuring various attributes. Measurement is important to many aspects of everyday life.

The measurement content domain is comprised of the following two main topic areas:

- Attributes and units
- Tools, techniques, and formulas

A measurable attribute is a characteristic of an object that can be quantified. For example, line segments have length, plane regions have area, and physical objects have mass. Learning about measurement begins with a realization of the need for comparison and the fact that different units are needed to measure

different attributes. The types of units that students use for measuring and the ways they use them should expand and shift as students move through the curriculum.

At both the fourth and eighth grades, age-appropriate performances expected of students include the use of instruments and tools to measure physical attributes, including length, area, volume, weight/mass, angle, temperature, and time, in standard or non-standard units and converting between systems of units. Students at the fourth grade are expected to use approximation and estimation, and simple formulas, to calculate areas and perimeters of squares and rectangles. At the eighth grade, the measurement domain is expanded to include the measurement of rate (such as speed or density) as well as the application of more complex formulas to measure compound areas and the surface areas of solids.

Measurement: Attributes and Units

Grade 4

- Use given non-standard units to measure length, area, volume, and time (e.g., paper clips for length, tiles for area, sugar cubes for volume).
 - Select appropriate standard units to measure length, area, mass/weight,* angle, and time (e.g., kilometers for car trips, centimeters for human height).
 - Use conversion factors between standard units (e.g., hours to minutes, grams to kilograms).
 - Recognize that total measures of length, area, volume, angle, and time do not change with position, decomposition into parts, or division.
- * More properly mass, but weight expressed in grams or kilograms is the common usage at these levels. Countries in which mass is the common usage for grades 4 and/or 8 will frame items accordingly.

Grade 8

- Select and use appropriate standard units to find measures of length, area, volume, perimeter, circumference, time, speed, density, angle, mass/weight.*
- Use relationships among units for conversions within systems of units, and for rates.

Measurement: Tools, Techniques, and Formulas

Grade 4

- Use instruments with linear or circular scales to measure length, weight, time, and temperature in problem situations (e.g., dimensions of a window, weight of a parcel).
- Estimate length, area, volume, weight, and time in problem situations (e.g., height of a building, volume of a block of material).
- Calculate areas and perimeters of squares and rectangles of given dimensions.
- Compute measurements in simple problem situations (e.g., elapsed time, change in temperature, difference in height or weight).

Grade 8

- Use standard tools to measure length, weight, time, speed, angle, and temperature in problem situations and to draw line segments, angles, and circles of given size.
- Estimate length, circumference, area, volume, weight, time, angle, and speed in problem situations (e.g., circumference of a wheel, speed of a runner).
- Compute with measurements in problem situations (e.g., add measures, find average speed on a trip, find population density).
- Select and use appropriate measurement formulas for perimeter of a rectangle, circumference of a circle, areas of plane figures (including circles), surface area and volume of rectangular solids, and rates.
- Find measures of irregular or compound areas by covering with grids or dissecting and rearranging pieces.
- Give and interpret information about precision of measurements (e.g., upper and lower bounds of a length reported as 8 centimeters to the nearest centimeter).



Geometry

Even at the fourth grade, the geometry content domain extends well beyond identification of geometric shapes. At both the fourth and eighth grades, students should be able to analyze the properties and characteristics of a variety of geometric figures, including lines, angles, and two- and three-dimensional shapes, and to provide explanations based on geometric relationships. The focus should be on geometric properties and their relationships. The geometry content area includes understanding coordinate representations and using spatial visualization skills to move between two- and three-dimensional shapes and their representations. Students should be able to use symmetry and apply transformation to analyze mathematical situations.

The major topic areas in geometry are:

- Lines and angles
- Two- and three-dimensional shapes
- Congruence and similarity
- Locations and spatial relationships
- Symmetry and transformations

Spatial sense is an integral component of the study and assessment of geometry. The cognitive range extends from making drawings and constructions to mathematical reasoning about

combinations of shapes and transformations. At both the fourth and eighth grades, students will be asked to describe, visualize, draw, and construct a variety of geometric figures, including angles, lines, triangles, quadrilaterals, and other polygons. Students should be able to combine, decompose, and analyze compound shapes. By the middle grades, they should be able to interpret or create top or side views of objects and use their understanding of similarity and congruence to solve problems. They should be able to make use of grids, find distances between points in the plane, and apply the Pythagorean theorem to solve real-world problems.

At both the fourth and eighth grades, students should be able to recognize line symmetry and draw symmetrical figures. They should be able to determine the effects of transformation. In the middle grades, students should understand and be able to describe rotations, translations, and reflections in mathematical terms (e.g., center, direction, and angle). As students progress through school, using proportional thinking in geometric contexts is important, as is making some initial links between geometry and algebra. Students should be able to solve problems using geometric models and explain relationships involving geometric concepts.

Geometry: Lines and Angles

Grade 4

- Classify angles as greater than, equal to, or less than a right angle (or 90°).
- Identify and describe parallel and perpendicular lines.
- Compare given angles and place them in order of size.

Grade 8

- Classify angles as acute, right, straight, obtuse, reflex, complementary, and supplementary.
- Recall the relationships for angles at a point, angles on a line, vertically opposite angles, angles associated with a transversal cutting parallel lines, and perpendicularity.
- Know and use the properties of angle bisectors and perpendicular bisectors of lines.



Geometry: Two- and Three-dimensional Shapes

Grade 4

- Know and use vocabulary associated with familiar two- and three-dimensional shapes.
- Identify common geometric shapes in the environment.
- Classify two- and three-dimensional shapes according to their properties.
- Know properties of geometric figures and use them to solve routine problems.
- Decompose shapes and rearrange the parts to form simpler shapes.

Grade 8

- Recall properties of geometric shapes: triangles (scalene, isosceles, equilateral, right-angled) and quadrilaterals (scalene, trapezoid, parallelogram, rectangle, rhombus, square).
- Use properties of familiar geometric shapes in a compound figure to make conjectures about properties of the compound figure.
- Recall properties of other polygons (regular pentagon, hexagon, octagon, decagon).
- Construct or draw triangles and rectangles of given dimensions.
- Apply geometric properties to solve routine and non-routine problems.
- Use Pythagorean theorem (not proof) to solve problems (e.g., find the length of a side of a right-angled triangle given the lengths of the other two sides; or, given the lengths of three sides of a triangle, determine whether the triangle is right-angled).

Geometry: Congruence and Similarity

Grade 4

- Identify triangles that have the same size and shape (congruent).
- Identify triangles that have the same shape but different sizes (similar).

Grade 8

- Identify congruent triangles and their corresponding measures.
- Identify congruent quadrilaterals and their corresponding measures.
- Consider the conditions of congruence to determine whether triangles with given corresponding measures (at least three) are congruent.
- Identify similar triangles and recall their properties.
- Use properties of congruence in mathematical and practical problem situations.
- Use properties of similarity in mathematical and practical problem situations.



Geometry: Locations and Spatial Relationships

Grade 4

- Use informal coordinate systems to locate points in a plane.
- Relate a net to the shape it will make.
- Recognize relationships between two-dimensional and three-dimensional shapes when shown nets and different two-dimensional views of three-dimensional objects.

Grade 8

- Locate points using number lines, coordinate grids, maps.
- Use ordered pairs, equations, intercepts, intersections, and gradient to locate points and lines in the Cartesian plane.
- Recognize relationships between two-dimensional and three-dimensional shapes when shown nets and different two-dimensional views of three-dimensional objects.

Geometry: Symmetry and Transformations

Grade 4

- Recognize line symmetry.
- Draw two-dimensional symmetrical figures.
- Recognize translation, reflection, and rotation.

Grade 8

- Recognize line and rotational symmetry for two-dimensional shapes.
- Draw two-dimensional symmetrical figures.
- Recognize, or demonstrate by sketching, translation, reflection, rotation, and enlargement.
- Use transformations to explain or establish geometric properties.

The data content domain includes understanding how to collect data, organize data that have been collected by oneself or others, and display data in graphs and charts that will be useful in answering questions that prompted the data collection. This content domain includes understanding issues related to misinterpretation of data (e.g., about recycling, conservation, or manufacturers' claims).

The data content domain consists of the following four major topic areas:

- Data collection and organization
- Data representation
- Data interpretation
- Uncertainty and probability

At the fourth and eighth grades, students can engage in simple data-gathering plans or work with data that have been gathered by others or generated by simulations. They should understand what various numbers, symbols, and points mean in data displays. For example, they should recognize that some numbers represent the values of the data and others represent the

frequency with which those values occur. Students should develop skill in representing their data, often using bar graphs, tables, or line graphs. They should be able to compare the relative merits of various types of displays.

Students at grades 4 and 8 should be able to describe and compare characteristics of data (shape, spread, and central tendency). They should be able to draw conclusions based on data displays. In the eighth grade, students should be able to identify trends in data, make predictions based on data, and evaluate the reasonableness of interpretations.

Probability will not be assessed at the fourth grade, and at the eighth grade the probability items will focus on assessing student understandings of the concept.⁴ By the eighth grade students' appreciation of probability should have increased beyond being able to designate the occurrence of familiar events as certain; as having greater, equal, or less likelihood; or as impossible, and they should be able to compute probabilities for simple events, or estimate probabilities from experimental data.

Data: Data Collection and Organization

Grade 4

- Match a set of data with appropriate characteristics of situations or contexts (e.g., outcomes from rolling a die).
- Organize a set of data by one characteristic (e.g., height, color, age, shape).

Grade 8

- Match a set of data, or a data display, with appropriate characteristics of situations or contexts (e.g., monthly sales of a product for a year).
- Organize a set of data by one or more characteristics using a tally chart, table, or graph.
- Recognize and describe possible sources of error in collecting and organizing data (e.g., bias, inappropriate grouping).
- Select the most appropriate data collection method (e.g., survey, experiment, questionnaire) to answer a given question, and justify the choice.

⁴ The original data representation, analysis, and probability category defined in the 1995 and 1999 assessments included a small number of items related to uncertainty and probability, but these depended heavily on knowledge of whole numbers, fractions, and proportionality.

Data: Data Representation**Grade 4**

- Read data directly from tables, pictographs, bar graphs, and pie charts.
- Display data using tables, pictographs, and bar graphs.
- Compare and match different representations of the same data.

Grade 8

- Read data from charts, tables, pictographs, bar graphs, pie charts, and line graphs.
- Display data using charts, tables, pictographs, bar graphs, pie charts, and line graphs.
- Compare and match different representations of the same data.

Data: Data Interpretation**Grade 4**

- Compare characteristics of related data sets (e.g., given data or representations of data on student heights in two classes, identify the class with the shortest/tallest person).
- Draw conclusions from data displays.

Grade 8

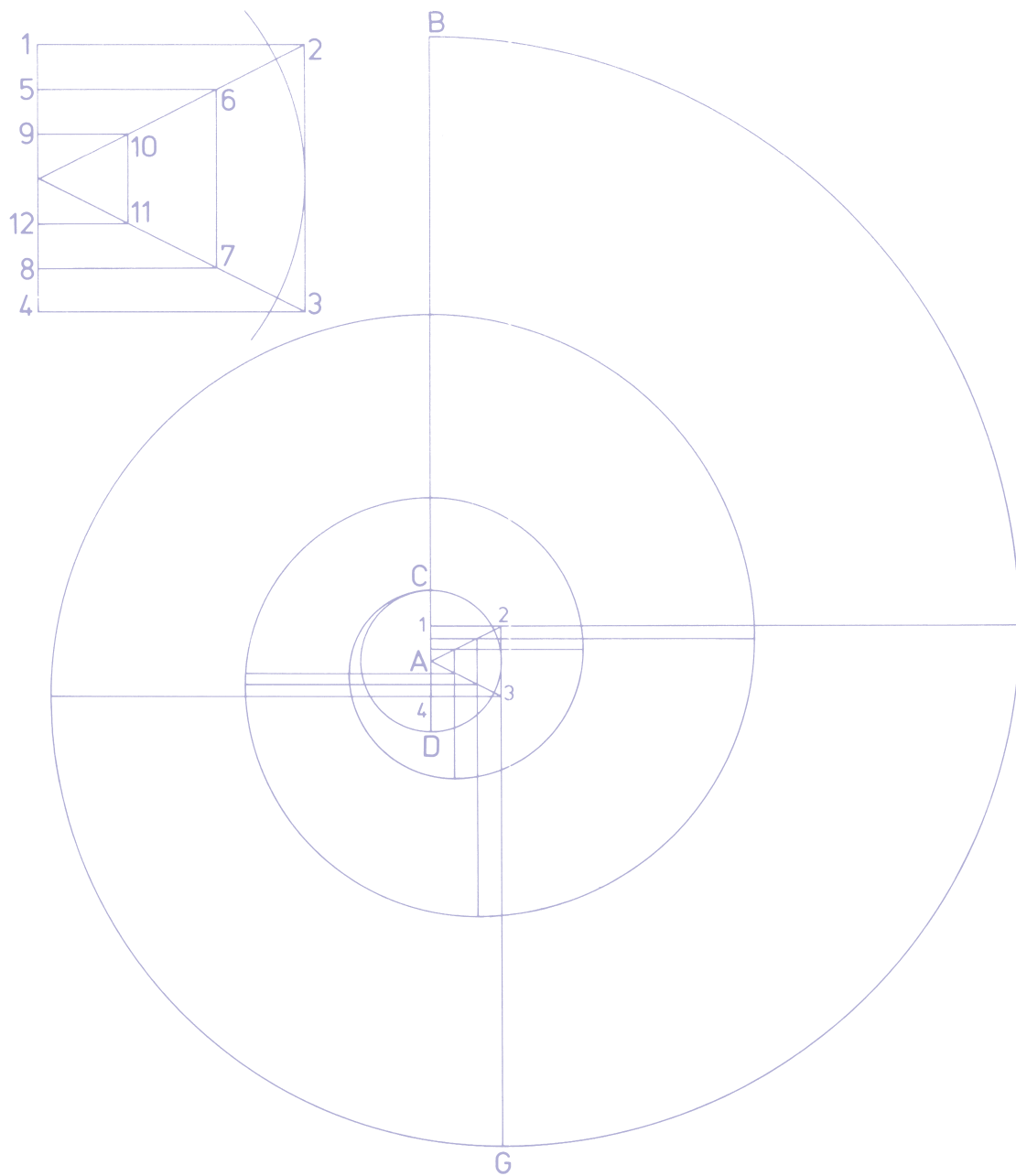
- Compare characteristics of data sets, using mean, median, range, and shape of distribution (in general terms).
- Interpret data sets (e.g., draw conclusions, make predictions, and estimate values between and beyond given data points).
- Evaluate interpretations of data with respect to correctness and completeness of interpretation.
- Use and interpret data sets to answer questions.

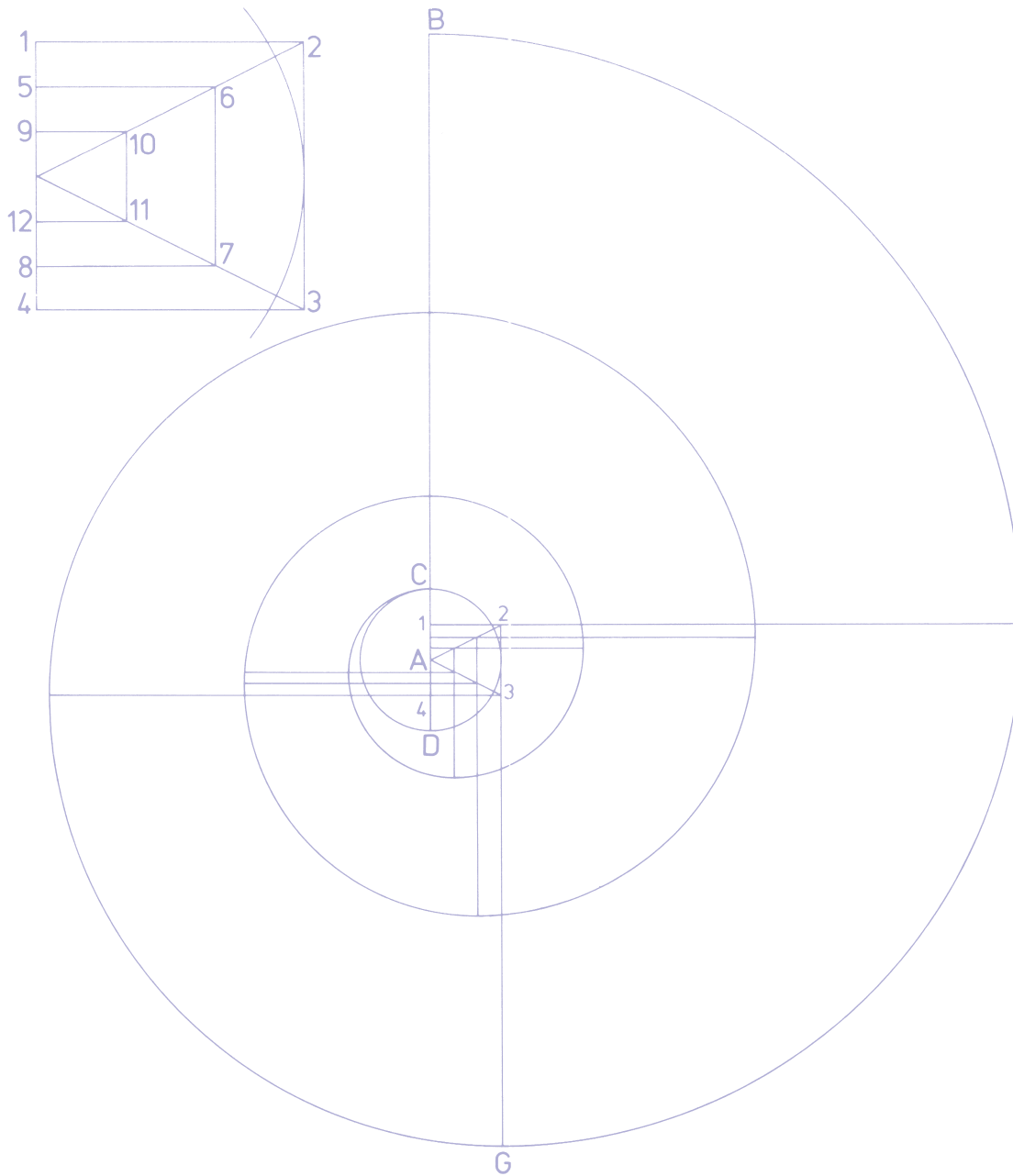
Data: Uncertainty and Probability**Grade 4**

- Not assessed.

Grade 8

- Judge the likelihood of an event as certain, more likely, equally likely, less likely, or impossible.
- Use data from experiments to estimate probabilities for favorable outcomes.
- Use problem conditions to calculate theoretical probabilities for possible outcomes.





Mathematics Cognitive Domains

To respond correctly to TIMSS test items, students will need to be familiar with the mathematics content of the items. Just as important, items will be designed to elicit the use of particular cognitive skills. Many of these skills and abilities are included with topics in the lists of assessable topics that comprise the content domains. However, as an aid in developing balanced tests in which appropriate weight is given to each cognitive domain across all topics, a full set of the learning outcomes mathematics educators would wish to see demonstrated is indispensable. Thus, descriptions of skills and abilities that make up the cognitive domains and will be assessed in conjunction with content are presented in the framework in some detail. These skills and abilities should play a central role in developing items and achieving a balance across the item sets at grades 4 and 8.

The student behaviors used to define the mathematics frameworks have been classified into the following four cognitive domains that are described in this section:

- Knowing Facts and Procedures
- Using Concepts
- Solving Routine Problems
- Reasoning

The specific student behaviors included in each cognitive domain comprise the outcomes sought by educational planners and practitioners around the world. Different groups within a society, and even among mathematics educators, have different views about the relative values of the cognitive skills, or at least about the relative emphases that should be accorded them in schools. For TIMSS they are all regarded as important, and a variety of test items will be used to measure each of them.

The skills and abilities included in each cognitive domain exemplify those that students could be expected to demonstrate in the TIMSS achievement tests. They are intended to apply to both grades 4 and 8, even though the accepted degree of sophistication in demonstrating behaviors will vary considerably between the two. The distribution of items over *knowing facts and procedures, using concepts, solving routine problems, and reasoning* also differs slightly between the two populations in accord with the mathematical experience of the target age/grade groups (see Exhibit 2).

As students' mathematical proficiency develops with the interaction of experience, instruction, and maturity, curricular emphasis moves from relatively straightforward problem situations to more complex tasks. In general, the cognitive complexity of tasks increases from one broad cognitive domain to the next. The intent is to allow for a progression from knowing a fact, procedure, or concept to using that knowledge to solve a problem, and from use of that knowledge in uncomplicated or familiar situations to the ability to engage in systematic reasoning.

Nevertheless, cognitive complexity should not be confused with item difficulty. For nearly all of the cognitive skills listed, it is possible to create relatively easy items as well as very challenging items. In developing items aligned with the skills, it is expected that a range of item difficulties will be obtained for each one, and that item difficulty should not affect the designation of the cognitive skill.

The following sections further describe the student behaviors, skills, and abilities used to define each cognitive domain with respect to the general capabilities expected of students. A table indicating specific behaviors to be elicited by items that are aligned with each skill in a domain follows the general descriptions, with examples of test items provided in some cases for illustration.





Knowing Facts and Procedures

Facility in using mathematics, or reasoning about mathematical situations, depends primarily on mathematical knowledge. The more relevant knowledge a student is able to recall, the greater the potential for engaging a wide range of problem-solving situations. Without access to a knowledge base that enables easy recall of the language and basic facts and conventions of number, symbolic representation, and spatial relations, students would find purposeful mathematical thinking impossible. *Facts* encompass the factual knowledge that provides the basic language of mathematics, and the essential mathematical facts and properties that form the foundation for mathematical thought.

Procedures form a bridge between more basic knowledge and the use of mathematics for solving routine problems, especially those encountered by many people in their daily lives. In essence a fluent use of procedures entails recall of sets of actions and how to carry them out. Students need to be efficient and accurate in using a variety of computational procedures and tools. They need to see that particular procedures can be used to solve entire classes of problems, not just individual problems.

Knowing Facts and Procedures

Recall	Recall definitions; vocabulary; units; number facts; number properties; properties of plane figures; mathematical conventions (e.g., algebraic notation such as $a \times b = ab$, $a + a + a = 3a$, $a \times a \times a = a^3$, $a/b = a \div b$).
Recognize/Identify	Recognize/identify mathematical entities that are mathematically equivalent, i.e., areas of parts of figures to represent fractions, equivalent familiar fractions, decimals, and percents; simplified algebraic expressions; differently oriented simple geometric figures.
Compute	Know algorithmic procedures for $+$, $-$, $\times$, $\div$, or a combination of these; know procedures for approximating numbers, estimating measures, solving equations, evaluating expressions and formulas, dividing a quantity in a given ratio, increasing or decreasing a quantity by a given percent. Simplify, factor, expand algebraic and numerical expressions; collect like terms.
Use Tools	Use mathematics and measuring instruments; read scales; draw lines, angles, or shapes to given specifications. Use straightedge and compass to construct the perpendicular bisector of a line, angle bisector, triangles, and quadrilaterals, given necessary measures.





Using Concepts

Familiarity with mathematical *concepts* is essential for the effective use of mathematics for problem solving, for reasoning, and thus for developing mathematical understanding.

Knowledge of concepts enables students to make connections between elements of knowledge that, at best, would otherwise be retained as isolated facts. It allows them to make exten-

sions beyond their existing knowledge, judge the validity of mathematical statements and methods, and create mathematical representations. Representation of ideas forms the core of mathematical thinking and communication, and the ability to create equivalent representations is fundamental to success in the subject.

Using Concepts

Know

Know that length, area, and volume are conserved under certain conditions; have an appreciation of concepts such as inclusion and exclusion, generality, equal likelihood, representation, proof, cardinality and ordinality, mathematical relationships, place value.

Grade 4 example: Decide whether area of paper is greater, the same, or less after a sheet of paper has been cut into strips (diagram shows complete sheet and separated strips).

Grade 8 example: Know that if five successive tosses of a fair coin come up "heads," the outcome of the next toss is as likely to be "tails" as "heads."

Classify

Classify/group objects, shapes, numbers, expressions, and ideas according to common properties; make correct decisions about class membership; order numbers and objects by attributes.

Grade 4 example: Select the triangles from a set of geometric figures having various shapes and numbers of sides.

Grade 8 example: Group pairs of quantities (lengths, weights, costs, etc.) in which the first quantity is greater than the second quantity.

Represent

Represent numbers using models; display mathematical information or data in diagrams, tables, charts, graphs; generate equivalent representations for a given mathematical entity or relationship.

Grade 4 example: Shade areas of figures to represent given fractions.

Grade 8 example: Given a function rule, generate ordered pairs that describe the function.

**Formulate**

Formulate problems or situations that could be modeled by given equations or expressions.

Grade 4 example: Jane has read 29 pages of her book. If the book has 87 pages, in the equation $87 - _ = 29$ the blank would contain the number of pages Jane still has to read. Make up another situation that this equation could be used for.

Grade 8 example: The equation $4x + 3 = 51$ could be used to solve the following problem: Four boxes are filled with golf balls, and 3 golf balls are left over. If there are 51 balls altogether, how many does each box hold? Make up a problem that the equation $25 - 3x = 1$ could be used to solve. (Do not solve the equation.)

Distinguish

Distinguish questions that can be addressed by given information, such as a data set, from those that cannot.

Grade 4 example: Given a bar graph, select from a set of questions those for which answers can be obtained from the graph.

Grade 8 example: The weights of boys in a class are <weights given>. Answers to which of the following questions can be found? What is the average weight of boys in the class? On average, do boys in the class weigh more than girls in the class? How many boys weigh more than 70 kg? What is the grade level of the class?



Solving Routine Problems

Students should be educated to recognize mathematics as an immense achievement of humanity, and to appreciate its nature. Nevertheless, mathematical knowledge for its own sake is probably not the most compelling reason for universal inclusion of mathematics in school curricula. Prime reasons for inclusion of mathematics are the increasing awareness that effectiveness as a citizen and success in the workplace are greatly enhanced by knowing and, more important, being able to use mathematics. The number of vocations that demand a high level of proficiency in the use of mathematics, or mathematical modes of thinking, has burgeoned with the advance of technology, and with modern management methods.

Problem solving is a central aim, and often means, of teaching school mathematics, and hence this and supporting skills (e.g., manipulate expressions, select, model, verify/check) feature prominently in the *solving routine problems* domain. In items aligned with this domain, the problem settings are more routine than those aligned with the reasoning domain. The routine problems will have been standard in classroom exercises designed to provide practice in particular methods or techniques. Some of these problems

will have been in words that set the problem situation in a quasi-real context. Solution of other such “textbook” type problems will involve extended knowledge of mathematical properties (e.g., solving equations). Though they range in difficulty, each of these types of “textbook” problems is expected to be sufficiently familiar to students that they will essentially involve selecting and applying learned procedures.

Problem solving is a desired outcome of mathematics instruction linked with many mathematics topics in the TIMSS framework. Problems may be set in real-life situations, or may be concerned with purely mathematical questions involving, for example, numeric or algebraic expressions, functions, equations, geometric figures, or statistical data sets. Therefore, problem solving has been included not only in solving routine problems but also in reasoning, depending on whether students are asked to solve routine problems or more non-routine problems (see following).

Solving Routine Problems

Select

Select/use an efficient method or strategy for solving problems where there is a known algorithm or method of solution, i.e., an algorithm or method students at the target level could be expected to be familiar with. Select appropriate algorithms, formulas, or units.

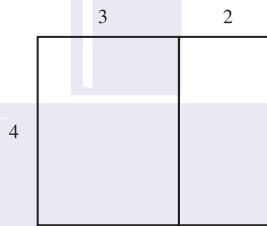
Grade 4 example: A class is presenting a class concert and the 28 members of the class each have 7 tickets to sell. To find the total number of tickets you should: divide 28 by 7; multiply 28 by 7; add 7 to 28; etc.

Grade 8 example: Given a problem that can be modeled by a simple equation, select the appropriate equation.



Model	Generate an appropriate model, such as an equation or diagram, for solving a routine problem.
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Interpret	Interpret given mathematical models (equations, diagrams, etc.); follow and execute a set of mathematical instructions. Grade 4 example: Given an unfamiliar (but not complex) figure or procedure, write the verbal instructions you would give to another student to have him/her reproduce the figure. Grade 8 example: Given a set of expressions including $4(3 + 2) = 4 \times 3 + 4 \times 2$, which one can the diagram be used to show?
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Apply	Apply knowledge of facts, procedures, and concepts to solve routine mathematical (including real-life) problems, i.e., problems similar to those target students are likely to have encountered in class.
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Verify/Check	Verify/check the correctness of the solution to a problem; evaluate the reasonableness of the solution to a problem. Grade 4 example: Mario estimates the area of a room in his house in square meters. His estimate is 1300 square meters. Is this likely to be a good estimate? Explain. Grade 8 example: Jack wants to find how far an airplane will travel in 3.5 hours at its top speed of 965 kph. He uses his calculator to multiply 3.5 by 965 and tells his friend Jenny that the answer is 33,775 km. Jenny says "that can't be right." How does she know?
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Reasoning

Reasoning mathematically involves the capacity for logical, systematic thinking. It includes intuitive and inductive reasoning based on patterns and regularities that can be used to arrive at solutions to non-routine problems. Non-routine problems are problems that are very likely to be unfamiliar to students. They make cognitive demands over and above those needed for solution of routine problems, even when the knowledge and skills required for their solution have been learned. Non-routine problems may be purely mathematical or may have real-life settings. Both types of items involve transfer of knowledge and skills to new situations, and interactions among reasoning skills are usually a feature.

Most of the other behaviors listed within the reasoning domain are those that may be drawn on in thinking about and solving such problems, but each by itself represents a valuable outcome of mathematics education, with the potential to influence learners' thinking more generally. For example, reasoning involves the ability to observe and make conjectures. It also involves making logical deductions based on specific assumptions and rules, and justifying results.

Reasoning

Hypothesize/Conjecture/ Predict	Make suitable conjectures while investigating patterns, discussing ideas, proposing models, examining data sets; specify an outcome (number, pattern, amount, transformation, etc.) that will result from some operation or experiment before it is performed. Grade 8 example: Twin primes are prime numbers with one other number between them. Thus, 5 and 7, 11 and 13, and 17 and 19 are pairs of twin primes. Make a conjecture about the numbers between twin primes.
Analyze	Determine and describe or use relationships between variables or objects in mathematical situations; analyze univariate statistical data; decompose geometric figures to simplify solving a problem; draw the net of a given unfamiliar solid; make valid inferences from given information.
Evaluate	Discuss and critically evaluate a mathematical idea, conjecture, problem solving strategy, method, proof, etc. Grade 4 example: Two painters use three tins of paint in painting a fence. They then have to use the same kind of paint to paint a similar fence that is twice as long and twice as high. One of the painters said that they would need about twice as much paint to paint the wall. Indicate whether the painter was right and support your answer with reasons. Grade 8 example: Comment on a survey with obvious flaws (too small a sample, non-representative sample, etc.).

**Generalize**

Extend the domain to which the result of mathematical thinking and problem solving is applicable by restating results in more general and more widely applicable terms.

Grade 4 example: Given the pattern 1, 4, 7, 10, ..., describe the relationship between each term and the next, and give the next term after 61.

Grade 8 example: Given that the sum of the angles of a triangle is 2 right angles, and given diagrams of 4-, 5-, and 6-sided polygons divided into triangles, describe the relationship between the number of sides of any polygon and the sum of its angles in right angles.

Connect

Connect new knowledge to existing knowledge; make connections between different elements of knowledge and related representations; make linkages between related mathematical ideas or objects.

Grade 8 example: A triangle ABC has sides $AB = 3\text{cm}$, $BC = 4\text{cm}$, and $CA = 5\text{cm}$. Which of these is the area of the triangle: 6 cm^2 , 7.5 cm^2 , 10 cm^2 , or 12 cm^2 ?

Synthesize/Integrate

Combine (disparate) mathematical procedures to establish results; combine results to produce a further result.

Grade 4 example: Solve a problem for which one of the key pieces of information must first be obtained from a table.

Grade 8 example: Combine results obtained from two distinct graphs to solve a problem.

Solve Non-routine Problems

Solve problems set in mathematical or real-life contexts where target students are very unlikely to have encountered closely similar items; apply mathematical procedures in unfamiliar contexts.

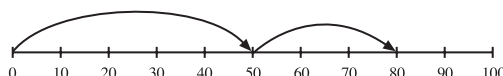
Grade 4 example: In a certain country the people write numbers as follows: 11 is written as $\nabla\nabla\Phi$, 42 is $\square\square\Phi\Phi$, and 26 is $\square\nabla\Phi$. How do people in this country write 37?

Grade 8 example: Given data and conditions in advertisements for competing products, select relevant data and find ways to make value comparisons valid in determining which product is most suitable in a particular context.

Justify/Prove

Provide evidence for the validity of an action or the truth of a statement by reference to mathematical results or properties; develop mathematical arguments to prove or disprove statements, given relevant information.

Grade 4 example: $50 + 30 = 80$. Use the number line below to show that this sentence is true. (Students to put in links or mark appropriately in some other way.)



Grade 8 example: Show that the sum of any two odd numbers is an even number.

Communicating Mathematically

Communicating mathematical ideas and processes is another set of skills that is seen as important for many aspects of living and fundamental to the teaching and learning of the subject. Representing, modeling, and interpreting, for example, are aspects of communication. While communication is an important outcome of mathematics education, it is not included as a separate cognitive domain. Rather, it may be thought of as an overarching dimension across all mathematics content areas and processes. Communication is fundamental to each of the other categories of *knowing facts and procedures*, *using concepts*, *solving routine problems*, and *reasoning*, and students' communication in and about mathematics should be regarded as assessable in each of these areas.

Students in TIMSS may demonstrate communication skills through description and explanation, such as describing or discussing a mathematical object, concept, or model. Communication also occurs in using mathematical terminology and notation; demonstrating the procedure used in simplifying, evaluating, or solving an equation; or using particular representational modes to present mathematical ideas. Communication is involved in explaining why a particular procedure or model has been used, and the reason for an unexpected or anomalous result. While describing and explaining are not explicitly listed as behaviors in the framework document, items across a wide range of content and processes will require these communication skills. Students could be asked to describe or explain why they chose a particular method, why they made a particular response, and so on.

Guidelines for Calculator Use

Although technology in the form of calculators and computers can help students learn mathematics, it should not be used to replace basic understanding and competencies. Like any teaching tool, calculators need to be used appropriately, and policies for their use differ across the TIMSS countries. Also, the availability of calculators varies widely. It would not be equitable to require calculator use when students in some countries may never have used them. Similarly, however, it is not equitable to deprive students of the use of a familiar tool. Thus, after considerable debate on this issue, beginning with TIMSS 2003 calculators are permitted but not required for newly developed grade 8 assessment materials.

The aim of the TIMSS guidelines for calculator use is to give students the best opportunity to operate in settings that mirror their classroom experience. Thus, if students are used to having calculators for their classroom activities, then the country should encourage students to use them during the assessment. On the other hand, if students are not used to having calculators or are not permitted to use them in their daily mathematics lessons, then the country need not permit their use. In developing the new assessment materials, every effort will be made to ensure that the test questions do not advantage or disadvantage students either way – with or without calculators.

Students at grade 4 will not be permitted to use calculators. Since calculators were not permitted at grade 8 in the 1995 and 1999 assessments, test administration procedures will ensure that they are not available for items from those assessments used to measure trends.